



Fracture Mechanics

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BASICS OF ELASTICITY

Stress–strain relations

Beginning with the definition of stress and strain tensors, σ and ε , respectively:

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}$$

Stress–strain relations

or in a vector (array) notation,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix}$$

where $\gamma_{ij} = 2\varepsilon_{ij}$ is the engineering shear strain component.

Stress–strain relations

In the seventeenth century, Hooke proposed his fundamental stress–strain relationship for one-dimensional linear elastic materials:

$$\sigma = E\varepsilon$$

where σ and ε are unidirectional stress and strain, respectively, and E is the elasticity (Young's) modulus.

The generalised Hooke's law for linear elastic materials can be defined as:

$$\boldsymbol{\sigma} = \mathbf{D} : \boldsymbol{\varepsilon}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the second order stress and strain tensors, respectively, and $\mathbf{D}$ is the fourth order elasticity modulus tensor with Cartesian components $\mathbf{D}_{ijkl}$.

Stress–strain relations

For general three dimensional problems, the fourth order elasticity tensor $\mathbf{D}$ can be represented by a two dimensional matrix $\mathbf{D}$ with components $\mathbf{D}_{\alpha\beta}, \alpha, \beta = 1-6$.

The stress-strain relation can now be re-written as,

$$\sigma_{ij} = \mathbf{D}_{ijkl} \varepsilon_{kl}$$

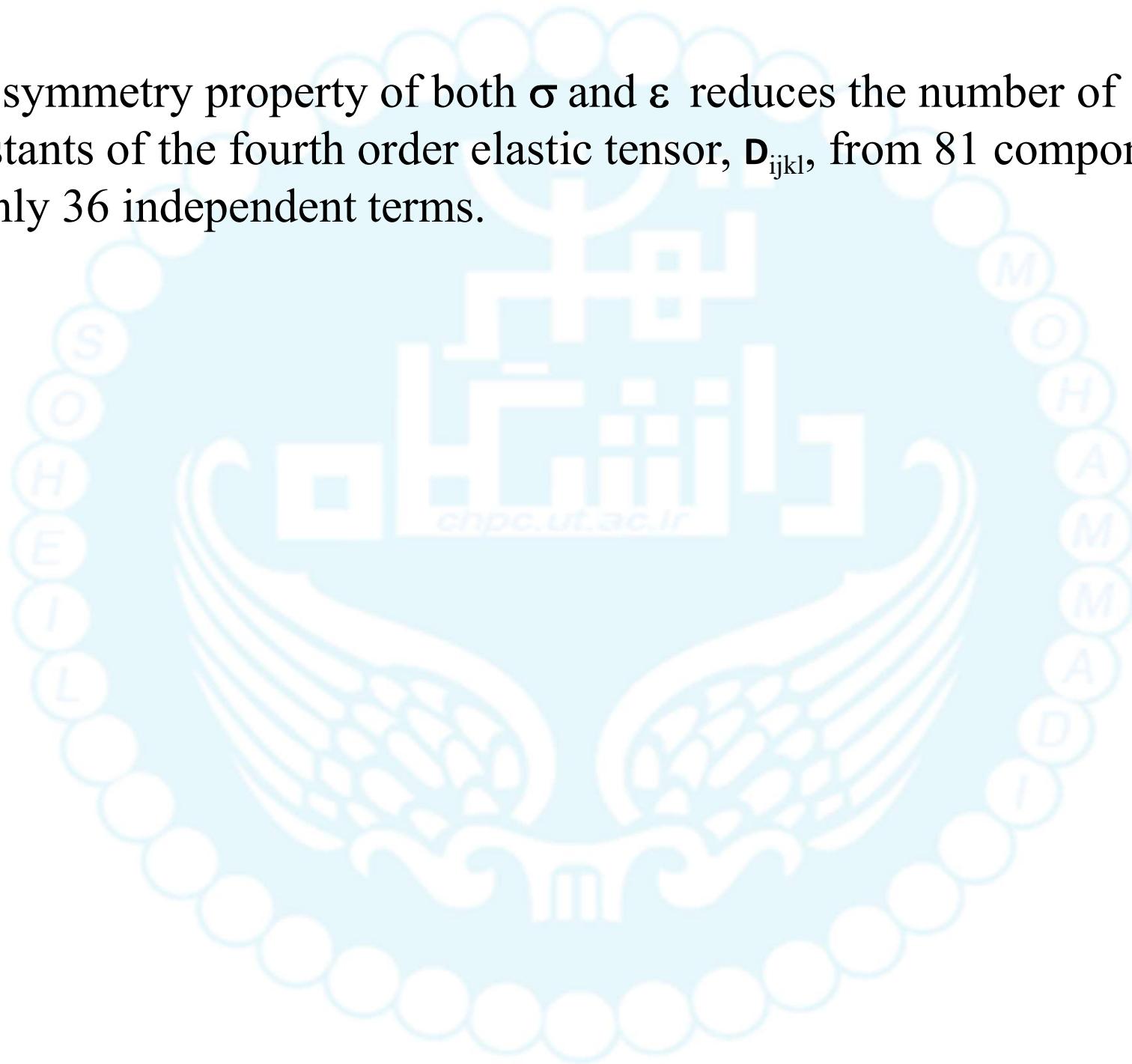
Similarly, the strain tensor can be defined in terms of the stress tensor,

$$\boldsymbol{\varepsilon} = \mathbf{C} : \boldsymbol{\sigma}$$

or in component form,

$$\varepsilon_{ij} = \mathbf{C}_{ijkl} \sigma_{kl}$$

The symmetry property of both σ and ε reduces the number of constants of the fourth order elastic tensor, $\mathbf{D}_{ijkl}$, from 81 components to only 36 independent terms.



For isotropic materials, the elasticity modulus can be determined from the bulk and shear modules, K and G , respectively, (Neto et al., 2008)

$$\mathbf{D} = 2G\bar{\mathbf{I}} + A(K - \frac{2}{3}G) \mathbf{I} \otimes \mathbf{I} = 2\mu \mathbf{I} + A\lambda \mathbf{I} \otimes \mathbf{I}$$

where $\mathbf{I}$ is the second-order identity tensor and $\mathbf{I}$ is the fourth-order symmetric identity tensor, and A

$$A = \begin{cases} 1 & \text{3D, plane strain, axisymmetric} \\ \frac{2G}{K + \frac{4}{3}G} = \frac{1-2\nu}{1-\nu} & \text{plane stress} \end{cases}$$

and λ and μ are the Lamé and shear modules

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}$$

$$\mu = \frac{E}{2(1 + \nu)}$$

Hooke's law for isotropic materials can also be written as:

$$\sigma_{ij} = \frac{E}{1+\nu} \left(\varepsilon_{ij} + \frac{\nu}{1-2\nu} \delta_{ij} \varepsilon_{kk} \right)$$

and in the form of strain–stress relations:

$$\varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \delta_{ij} \sigma_{kk}$$

where δ_{ij} is the Kronecker delta function.

3D isotropic materials

The expanded form for 3D isotropic materials can then be written as:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$$

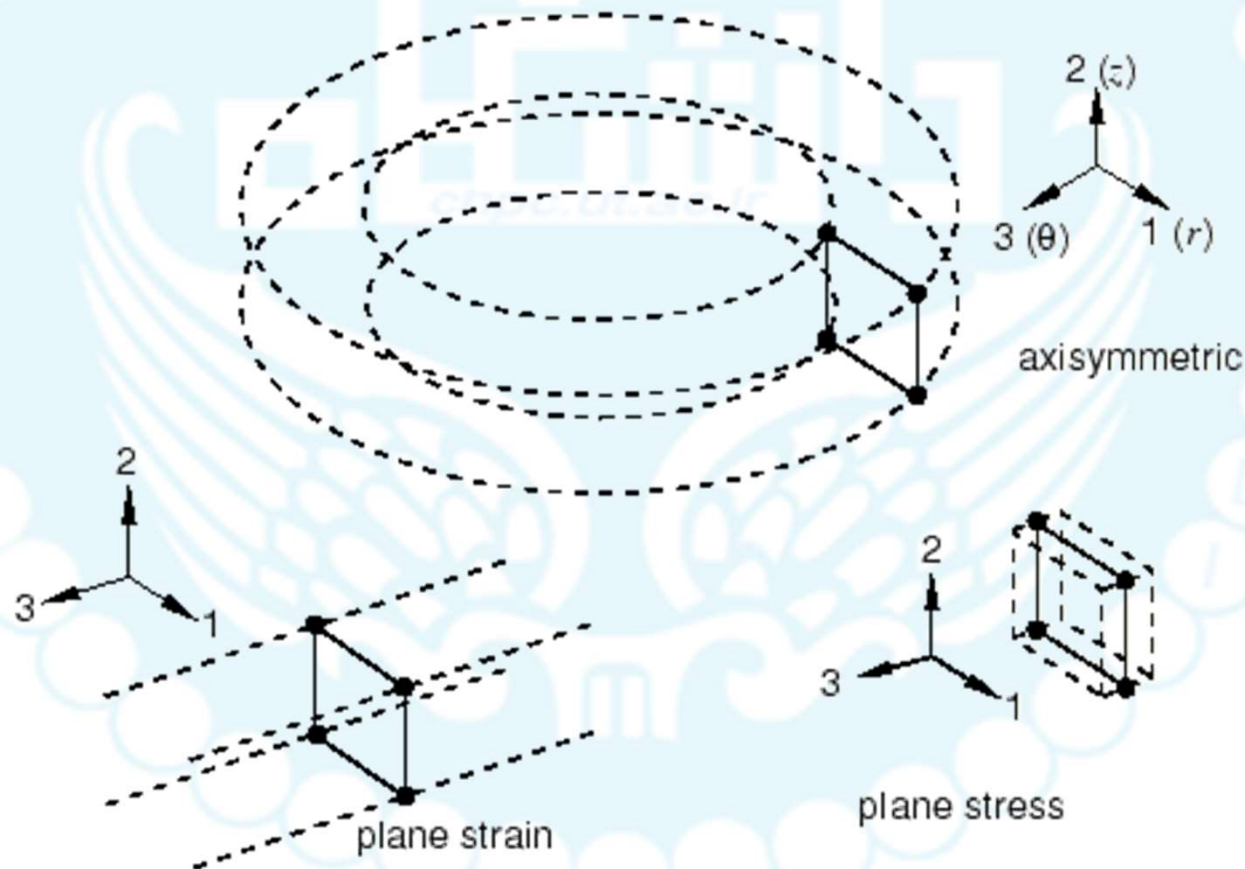
BASICS OF ELASTICITY

or

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix}$$

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These equations are further simplified for the plane stress, plane strain and two-dimensional axisymmetric cases.



Plane stress

For the plane stress case, the dimension in one direction (thickness) is neglected compared to the two others, and

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

Where $\gamma_{xy} = 2\varepsilon_{xy}$ is the engineering shear strain component. The strain component along the thickness direction, ε_{zz} , can also be derived from the Poisson's effect:

$$\varepsilon_{zz} = -\frac{\nu}{1-\nu} (\varepsilon_{xx} + \varepsilon_{yy})$$

Plane strain

A plane strain case resembles a long body undergoing no variation in load or geometry in the longitudinal direction. As a result,

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ \nu & \nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

Here, some other useful definitions are presented. First, the effective Young's modulus, E' , is defined as:

$$E' = \begin{cases} E & \text{plane stress} \\ \frac{E}{1-\nu^2} & \text{plane strain} \end{cases}$$

Axisymmetric

For an axisymmetric solid of revolution, a form almost similar to the plane strain relation is derived in terms of polar coordinate systems:

$$\begin{Bmatrix} \sigma_{zz} \\ \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{zz} \\ \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \gamma_{rz} \end{Bmatrix}$$

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} \quad \varepsilon_{\theta\theta} = \frac{u}{r} \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

Here, some other useful definitions are presented.

First, the effective Young's modulus, E' , is defined as:

$$E' = \begin{cases} E & \text{plane stress} \\ \frac{E}{1-\nu^2} & \text{plane strain} \end{cases}$$

and another frequently used parameter κ

$$\kappa = \begin{cases} \frac{3-\nu}{1+\nu} & \text{plane stress} \\ 3-4\nu & \text{plane strain} \end{cases}$$

Strain energy density

When an elastic body is subjected to a loading, the body undergoes displacements which are linearly related to the applied loading. As a result, a certain amount of work is stored in the body in the form of potential strain energy. The strain energy density can be defined as:

$$U_s = \int \sigma_{ij} d\varepsilon_{ij} = \frac{1}{2} \sigma^T \varepsilon$$

The strain energy will be directly used in the definitions of a number of fundamental concepts of fracture mechanics, such as the J integral.

Airy stress function

Airy developed the idea of a stress function which can satisfy both the equilibrium and the compatibility conditions (Sharifabadi 1990).

He showed that in absence of body forces, such a function Φ must satisfy the Laplace equation:

$$\nabla^4 \Phi = \nabla^2 (\nabla^2 \Phi) = 0$$

Airy stress function

In Cartesian coordinates:

$$\begin{aligned}\nabla^4 \Phi &= \nabla^2 (\nabla^2 \Phi) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) \\ &= \frac{\partial^4 \Phi}{\partial x^4} + 2 \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0\end{aligned}$$

and in Polar coordinates:

$$\nabla^4 \Phi = \nabla^2 (\nabla^2 \Phi) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \right) = 0$$

The stress components can be obtained from derivatives of the stress function Φ :

Cartesian coordinates:

$$\left\{ \begin{array}{l} \sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2} \\ \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} \\ \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \end{array} \right.$$

Polar coordinates:

$$\left\{ \begin{array}{l} \sigma_{rr} = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \\ \sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2} \\ \sigma_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right) \end{array} \right.$$

The Laplace equation can then be written as:

$$\nabla^4 \Phi = \nabla^2 (\sigma_{xx} + \sigma_{yy}) = 0$$

or in polar coordinates:

$$\nabla^4 \Phi = \nabla^2 (\sigma_{rr} + \sigma_{\theta\theta}) = 0$$

The Airy function must also satisfy the natural (stress) boundary conditions, which further limits its application for complex problems.

Complex stress functions

Kolonov and Muskheshvili (1953) developed the idea of complex stress functions, which enables finding solutions for more general problems including sharp corners, cracks and openings.

Assuming $\psi(z)$ and $\chi(z)$ to be two harmonic analytic functions of z ,

$$\nabla^2(\psi) = \nabla^2(\chi) = 0$$

$$z = x + iy = re^{i\theta}$$

Any stress function Φ can be expressed as:

$$\Phi = \text{Re} [\bar{z} \psi(z) + \chi(z)]$$

where

$$\bar{z} = x - iy$$

The three stress components σ_{xx} , σ_{yy} and σ_{xy} can be solved from:

$$\sigma_{xx} + \sigma_{yy} = 4 \operatorname{Re} [\psi'(z)]$$

$$-\sigma_{xx} + \sigma_{yy} + 2i\sigma_{xy} = 2[\bar{z}\psi''(z) + \chi''(z)]$$

where ' and “ denote the first and second derivatives of a function, respectively.

Cartesian displacements u_x and u_y can also be expressed in terms of the complex functions:

$$u_x + iu_y = \frac{3-\nu}{E}\psi(z) - \frac{1+\nu}{E}[\bar{z}\psi'(\bar{z}) + \chi'(\bar{z})]$$

Curvilinear coordinates

A Cartesian (x,y) or polar coordinate (r, θ) systems are usually used to define the position of a point. In general, however, any set of independent variables (α, β) can be used to define the position of any node; representing a general curvilinear coordinate system.

A point in the Cartesian system can be defined by:

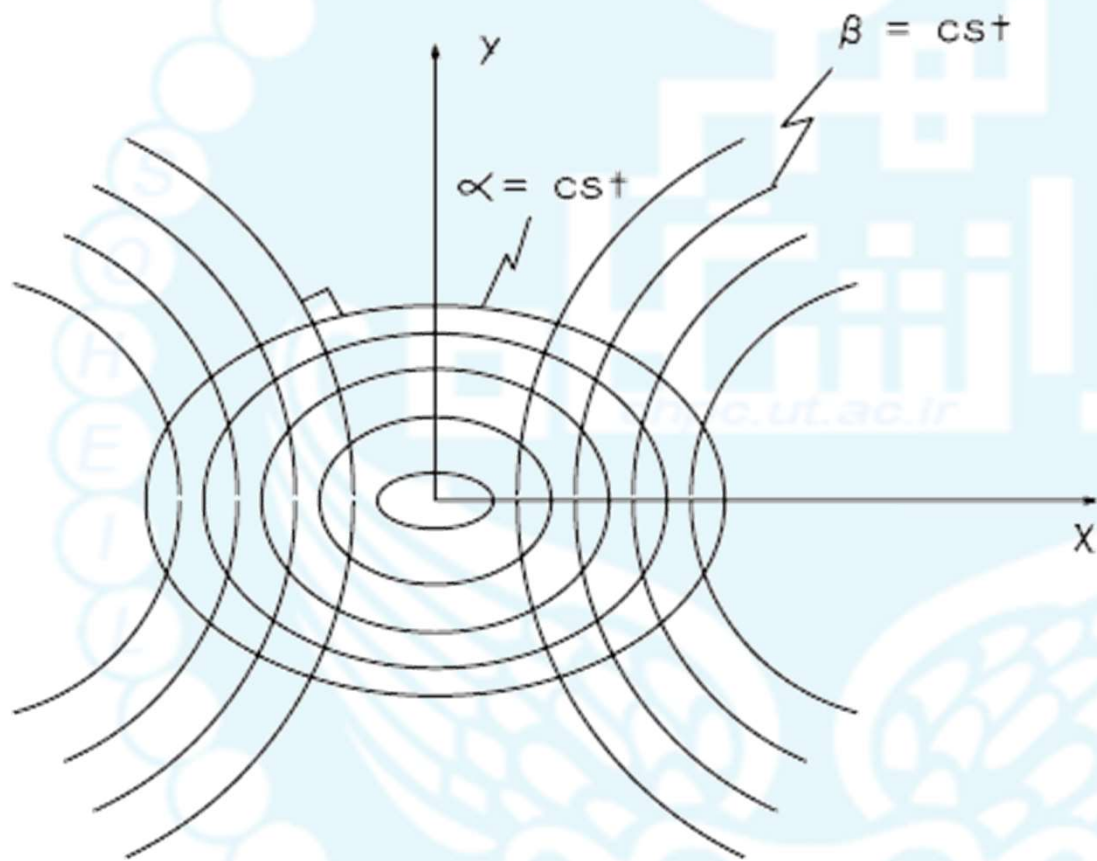
$$z = x + iy$$

and in the curvilinear system:

$$p = \alpha + i\beta$$

Different curvilinear systems are defined based on the way p and z are related.

Curvilinear coordinates



$$z = x + iy$$

$$p = \alpha + i\beta$$

Curvilinear coordinates

One of the most frequently used curvilinear system is described here.

Assume the following equation relates p and z :

$$z = c \cosh p, \quad \begin{cases} z = x + iy \\ p = \alpha + i\beta \end{cases}$$

where c is a constant.

recalling:

$$\cosh \alpha = \frac{1}{2} (e^{\alpha} + e^{-\alpha})$$

$$\sinh \alpha = \frac{1}{2} (e^{\alpha} - e^{-\alpha})$$

$$e^{i\beta} = \cos \beta + i \sin \beta$$

Curvilinear coordinates

Then,

$$\begin{aligned} z = x + iy &= \frac{c}{2} \left(e^{\alpha + i\beta} + e^{-\alpha - i\beta} \right) \\ &= \dots \\ &= c \left(\cosh \alpha \cos \beta + i \sinh \alpha \sin \beta \right) \end{aligned}$$

Separating the real and imaginary parts:

$$x = c \cosh \alpha \cos \beta = \frac{c}{2} \cos \beta \left(e^{\alpha} + e^{-\alpha} \right)$$

$$y = c \sinh \alpha \sin \beta = \frac{c}{2} \sin \beta \left(e^{\alpha} - e^{-\alpha} \right)$$

Curvilinear coordinates

Eliminating β , leads to:

$$\frac{x^2}{\cosh^2 \alpha} + \frac{y^2}{\sinh^2 \alpha} = c^2$$

which is an ellipse in the Cartesian system for any constant α .

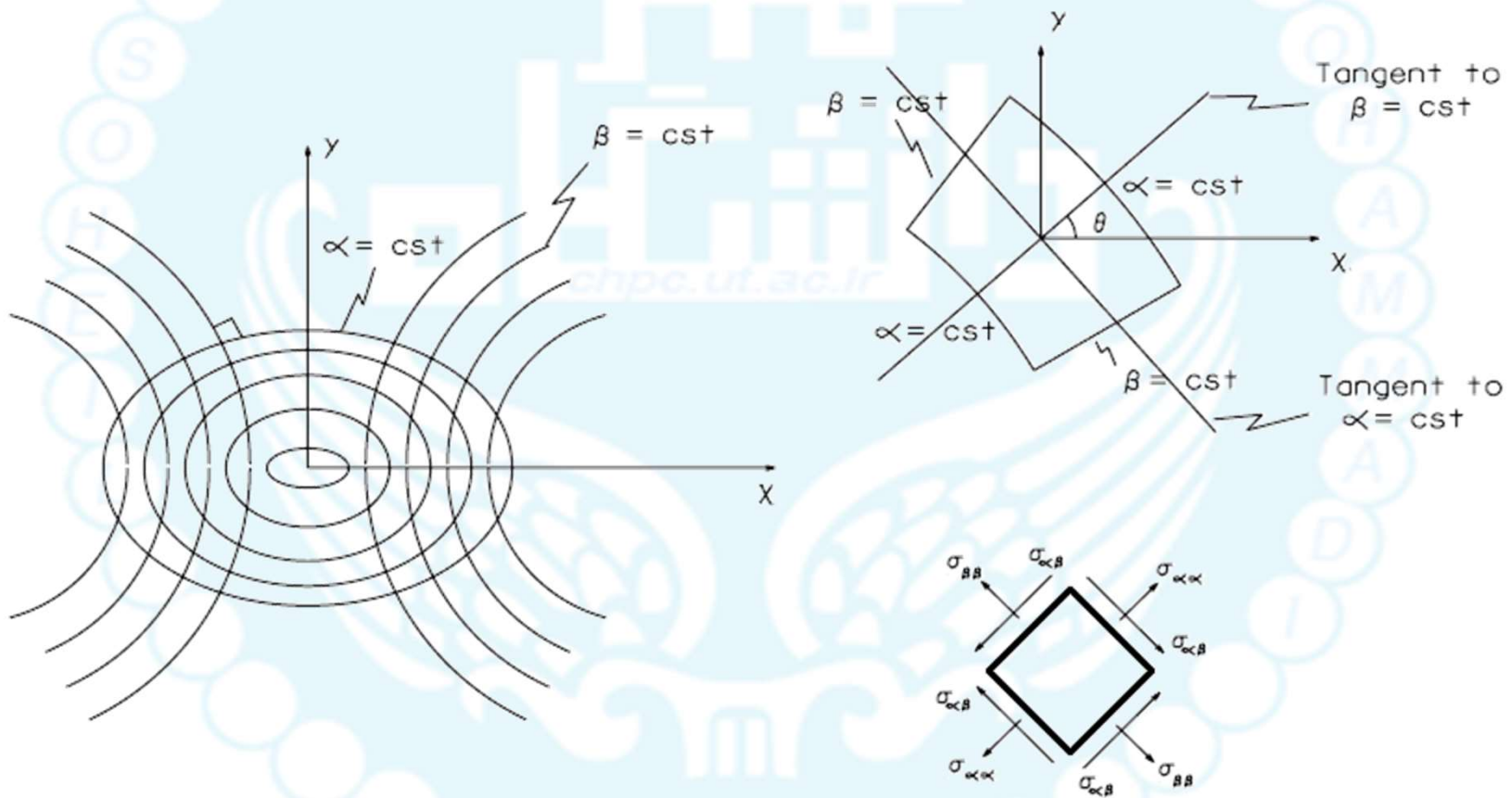
Similarly, eliminating α results in:

$$\frac{x^2}{\cos^2 \beta} - \frac{y^2}{\sin^2 \beta} = c^2$$

which is a hyperbolic in the Cartesian system for any constant β , which intersects the ellipse at the right angle (they are orthogonal).

Curvilinear coordinates

$$z = c \cosh p$$



Curvilinear stress functions

The stress components in terms of curvilinear coordinates can then be defined as:

$$\sigma_{\alpha\alpha} + \sigma_{\beta\beta} = 4\operatorname{Re} [\psi'(z)]$$

$$-\sigma_{\alpha\alpha} + \sigma_{\beta\beta} + 2i\sigma_{\alpha\beta} = 2[\bar{z}\psi''(z) + \chi''(z)] e^{2i\theta}$$

and for displacements

$$2\mu(u_{\alpha} - iu_{\beta}) = \left[\frac{3-\nu}{1+\nu} \psi(z) - z\psi'(\bar{z}) - \chi'(\bar{z}) \right] e^{-i\theta}$$

The special case of polar coordinates is obtained by setting:

$$\alpha \rightarrow r = \sqrt{x^2 + y^2}$$

$$\beta \rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

It should be noted that the present form of the stress function can be further generalised as:

$$\Phi = \text{Re} [\bar{z} \psi(z) + \chi(z)]$$

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$$\Phi = \bar{z} \psi(z) + \chi(z)$$

